

A note on the one-dimensional hydrogen atom with minimal length uncertainty

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Abstract. We present exact energy spectrum and eigenfunctions of the one-dimensional hydrogen atom in the presence of the minimal length uncertainty. By requiring the self-adjointness property of the Hamiltonian, we completely determine the quantization condition. We indicate that the single-valuedness criteria of the eigenfunctions in non-deformed case is an emergent condition and the semiclassical solutions exactly coincide with the quantum mechanical results. The behavior of the wave functions at the origin in coordinate space and in quasiposition space is discussed finally.

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1. Introduction

Studying the effects of the generalized (gravitational) uncertainty principle (GUP) on various physical systems has attracted much attention in recent years and many papers have appeared in literature to address the modification of the Hamiltonians and their energy spectrum and eigenfunctions in the presence of the minimal length uncertainty. Indeed, this idea arises naturally from various candidates of quantum gravity such as string theory [1–6], loop quantum gravity [7], noncommutative spacetime [8–10], and black-holes gedanken experiments [11,12]. All these studies imply a finite lower bound to the possible resolution of length of the order of the Planck length $\ell_{Pl} = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-35}m$ where G is Newton's gravitational constant.

The problem of the hydrogen atom is studied in [13–17] and the exact energy eigenvalues and eigenfunctions are obtained. In the presence of the minimal length, Akhoury and Yao [18], and Bouaziz and Ferkous [19] have solved this problem for zero angular momentum states and found exact expressions for the GUP-corrected solutions. This problem is also studied numerically and perturbatively in Refs. [20,21]. Fityo *et al* detected a single-valuedness problem in Ref. [18] and tried to present the quantization condition in one-dimension by requiring the symmetricity of the inverse of the position

operator on the eigenfunctions [22]. However, since there is a free parameter in their solution, the energy spectrum is not completely determined.

Here, using an alternative representation of the deformed algebra, we find the quantization condition by requiring the self-adjointness property of the Hamiltonian. Also we show that the validity of the single-valuedness criteria for one-dimensional hydrogen atom in ordinary quantum mechanics is an emergent condition. The quasiposition space solutions, coordinate space solutions, semiclassical solutions, and the validity of WKB approximation are discussed finally.

2. The generalized uncertainty principle

Consider the following one-dimensional deformed commutation relation

$$[X, P] = i\hbar(1 + \beta P^2), \quad (1)$$

where for $\beta = 0$ we recover the well-known commutation relation in ordinary quantum mechanics. To exactly satisfy the above algebra, Kempf, Mangano and Mann (KMM) have proposed the following representation [9]:

$$X = (1 + \beta p^2)x, \quad (2)$$

$$P = p, \quad (3)$$

where x and p are canonical position and momentum operators, i.e., $[x, p] = i\hbar$. In fact, X and P are symmetric operators on the dense domain S_∞ with respect to the following scalar product:

$$\langle \psi | \phi \rangle = \int_{-\infty}^{+\infty} \frac{dp}{1 + \beta p^2} \psi^*(p) \phi(p). \quad (4)$$

However, this representation is not unique and using appropriate canonical transformations, we can obtain an alternative representation. For instance, consider the following exact representation [23, 24]:

$$X = x, \quad (5)$$

$$P = \frac{\tan(\sqrt{\beta}p)}{\sqrt{\beta}}. \quad (6)$$

These operators are formally self-adjoint, i.e. $A = A^\dagger$ for $A \in \{X, P\}$ and they are symmetric subject to the inner product

$$\langle \psi | \phi \rangle = \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} dp \psi^*(p) \phi(p). \quad (7)$$

Moreover, the ordinary nature of the position operator is preserved in this representation. In fact, both exact representations are equivalent and they are related by the following canonical transformation:

$$X \longrightarrow \left[1 + \arctan^2(\sqrt{\beta}P) \right] X, \quad (8)$$

$$P \longrightarrow \arctan(\sqrt{\beta}P) / \sqrt{\beta}, \quad (9)$$

which transforms equations (5) and (6) to equations (2) and (3) subjected to equation (1). In this representation, the completeness relation and scalar product can be written as

$$\langle p'|p\rangle = \delta(p - p'), \quad (10)$$

$$\int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} dp |p\rangle\langle p| = 1. \quad (11)$$

Note that the operator X is not a true self-adjoint operator. Although its adjoint $X^\dagger = i\hbar\partial/\partial p$ has the same formal expression, it acts on a different space of functions, namely

$$\mathcal{D}(X) = \left\{ \phi, X\phi \in \mathcal{L}^2\left(\frac{-\pi}{2\sqrt{\beta}}, \frac{+\pi}{2\sqrt{\beta}}\right); \phi\left(\frac{+\pi}{2\sqrt{\beta}}\right) = \phi\left(\frac{-\pi}{2\sqrt{\beta}}\right) = 0 \right\}, \quad (12)$$

$$\mathcal{D}(X^\dagger) = \left\{ \varphi, X^\dagger\varphi \in \mathcal{L}^2\left(\frac{-\pi}{2\sqrt{\beta}}, \frac{+\pi}{2\sqrt{\beta}}\right); \text{no other restriction on } \varphi \right\}. \quad (13)$$

To show this let us check the symmetricity of the position operator as

$$\langle X\phi | \phi \rangle = \langle \phi | X\phi \rangle, \quad (14)$$

where ϕ and φ belong to the domain of X and X^\dagger , respectively. This condition can be rewritten as

$$\begin{aligned} & -i \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \frac{\partial \varphi^*(p)}{\partial p} \phi(p) dp - i \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \varphi^*(p) \frac{\partial \phi(p)}{\partial p} dp \\ & = -i\varphi^*(p)\phi(p)\Big|_{p=\frac{+\pi}{2\sqrt{\beta}}} + i\varphi^*(p)\phi(p)\Big|_{p=\frac{-\pi}{2\sqrt{\beta}}} = 0, \end{aligned} \quad (15)$$

which verifies (12) and (13). As it is also shown in [25], any operator X which obeys the commutation relation (1) is not a true self-adjoint operator. Before proceed further, let us categorize the operators in terms of their self-adjointness properties as follows:

- (i) Self-adjoint operators: $A = A^\dagger$ and $\mathcal{D}(A) = \mathcal{D}(A^\dagger)$.
- (ii) Symmetric operators: $\langle A\psi | \phi \rangle = \langle \psi | A\phi \rangle$ for all $\psi, \phi \in \mathcal{D}(A)$.

Note that a self-adjoint operator is symmetric but its inverse is not always true. However, the symmetricity of an operator is sufficient to ensure the reality of the eigenvalues.

3. Momentum representation

Now consider the one-dimensional hydrogen atom eigenvalue problem:

$$P^2\phi - \frac{\alpha}{X}\phi = E\phi, \quad (16)$$

where we set $\hbar = 1 = 2m$ and we take $E = -\epsilon$. In this representation we have $X\phi = i\frac{\partial\phi}{\partial p}$ and the action of inverse operator $1/X$ is expressed as

$$\frac{1}{X}\phi(p) = -i \int_{-\frac{\pi}{2\sqrt{\beta}}}^p \phi(q) dq + c, \quad \frac{-\pi}{2\sqrt{\beta}} < p < \frac{+\pi}{2\sqrt{\beta}}, \quad (17)$$

where, as we show below, c should be a constant. Note that the application of (17) with $c = 0$ leads to the existence of the only trivial solution $\phi(p) = 0$ [22]. Also in the absence of GUP the presence of c corresponds to derivative discontinuity of eigenfunctions at the origin in the coordinate representation [14]. By definition (17) we obtain

$$X \frac{1}{X} \phi = \phi, \quad (18)$$

$$\frac{1}{X} X \phi = \phi + c, \quad (19)$$

and

$$\left[X, \frac{1}{X} \right] \phi = -c. \quad (20)$$

Similarly, since the adjoint of the position operator has the same formal expression of the position operator, i.e. $X^\dagger \phi = i \frac{\partial \phi}{\partial p}$, the action of the adjoint of $1/X$ is expressed as

$$\left(\frac{1}{X} \right)^\dagger \phi(p) = -i \int_{-\frac{\pi}{2\sqrt{\beta}}}^p \phi(q) dq + c^*, \quad \frac{-\pi}{2\sqrt{\beta}} < p < \frac{+\pi}{2\sqrt{\beta}}, \quad (21)$$

and therefore we find

$$X^\dagger \left(\frac{1}{X} \right)^\dagger \phi = \phi, \quad (22)$$

$$\left(\frac{1}{X} \right)^\dagger X^\dagger \phi = \phi + c^*, \quad (23)$$

$$\left[X^\dagger, \left(\frac{1}{X} \right)^\dagger \right] \phi = -c^*. \quad (24)$$

We now prove that X^{-1} is not a linear operator. In a basis which X as a linear operator is diagonal, the formal operational relation $X \frac{1}{X} = 1$ (18) implies that if X^{-1} is a linear operator with a matrix representation, it is also diagonal. So we have

$$[X, X^{-1}] = 0, \quad (25)$$

which apparently contradicts with equation (20). The same argument also applies for X^\dagger . Explicitly we have

$$\begin{aligned} \frac{1}{X} [\mu \phi(p) + \nu \varphi(p)] &= -i\mu \int_{-\frac{\pi}{2\sqrt{\beta}}}^p \phi(q) dq - i\nu \int_{-\frac{\pi}{2\sqrt{\beta}}}^p \varphi(q) dq + c, \\ &\neq \mu \frac{1}{X} \phi(p) + \nu \frac{1}{X} \varphi(p), \end{aligned} \quad (26)$$

and a similar relation for $\left(\frac{1}{X} \right)^\dagger$. Thus c is a constant not a linear functional.

Equations (17) and (21) now result in

$$\left[\frac{1}{X} - \left(\frac{1}{X} \right)^\dagger \right] \phi = 2\text{Im}[c]. \quad (27)$$

Notice that this relation is valid regardless of the actual nature of c . Now since the momentum operator P is Hermitian, i.e. $P = P^\dagger$, the Hermiticity nature of the Hamiltonian requires $\frac{1}{X} = \left(\frac{1}{X} \right)^\dagger$ or

$$\text{Im}[c] = 0. \quad (28)$$

As we shall see this condition completely determines the quantization condition. Indeed, the presence of a free parameter in the energy spectrum reported by Fityo *et al* [22] is due to the fact that they only implemented the symmetricity condition for the Hamiltonian. Here, we impose an stronger condition i.e. “self-adjointness” to fix the energy spectrum. To ensure the self-adjointness of the Hamiltonian we will check the domains of the Hamiltonian and its adjoint at the end of this section.

The Schrödinger equation in momentum space then reads

$$-\frac{\tan^2(\sqrt{\beta}p)}{\beta}\phi(p) - i\alpha \int_{-\frac{\pi}{2\sqrt{\beta}}}^p \phi(q) dq + \alpha c = \epsilon \phi(p). \quad (29)$$

If we differentiate this equation with respect to p we find

$$\phi'(p) + \beta \frac{\frac{2\sec^2(\sqrt{\beta}p)\tan(\sqrt{\beta}p)}{\sqrt{\beta}} + i\alpha}{\tan^2(\sqrt{\beta}p) + \beta\epsilon} \phi(p) = 0. \quad (30)$$

The solution is

$$\phi(p) = \mathcal{A} \frac{2\beta\epsilon \cos^2(\sqrt{\beta}p)}{1 + \beta\epsilon - (1 - \beta\epsilon) \cos(2\sqrt{\beta}p)} \exp \left[\frac{i\alpha}{1 - \beta\epsilon} \left(\beta p - \frac{1}{\sqrt{\epsilon}} \arctan \left[\frac{\tan(\sqrt{\beta}p)}{\sqrt{\beta\epsilon}} \right] \right) \right], \quad (31)$$

where can be rewritten as

$$\phi(p) = \mathcal{A} \frac{2\beta\epsilon \cos^2(\sqrt{\beta}p)}{1 + \beta\epsilon - (1 - \beta\epsilon) \cos(2\sqrt{\beta}p)} \exp \left[\frac{i\alpha\beta}{1 - \beta\epsilon} p \right] \left[\frac{1 - i \frac{\tan(\sqrt{\beta}p)}{\sqrt{\beta\epsilon}}}{1 + i \frac{\tan(\sqrt{\beta}p)}{\sqrt{\beta\epsilon}}} \right]^{\frac{\alpha}{2\sqrt{\epsilon}(1 - \beta\epsilon)}}, \quad (32)$$

where \mathcal{A} is the normalization coefficient. Substituting the expression for eigenfunctions (32) into the eigenvalue equation (29) results in

$$c = \frac{1}{\alpha} \lim_{p \rightarrow -\frac{\pi}{2\sqrt{\beta}}} \left(\frac{\tan^2(\sqrt{\beta}p)}{\beta} + \epsilon \right) \phi(p) = \mathcal{A} \frac{\epsilon}{\alpha} e^{\frac{i\pi\alpha}{2\sqrt{\epsilon}(1 + \sqrt{\beta\epsilon})}}. \quad (33)$$

So the probability density in momentum space reads

$$|\phi(p)|^2 = \left[\frac{2\mathcal{A}\beta\epsilon \cos^2(\sqrt{\beta}p)}{1 + \beta\epsilon - (1 - \beta\epsilon) \cos(2\sqrt{\beta}p)} \right]^2. \quad (34)$$

Also the normalization coefficient is given by

$$\mathcal{A} = \sqrt{\frac{2}{\pi}} \epsilon^{-1/4} \frac{1 + \sqrt{\beta\epsilon}}{\sqrt{(1 + 2\sqrt{\beta\epsilon})}}. \quad (35)$$

The Hermiticity condition (28) now implies

$$\sin \left[\frac{\pi\alpha}{2\sqrt{\epsilon}(1 + \sqrt{\beta\epsilon})} \right] = 0, \quad (36)$$

which results in the following quantization condition:

$$\frac{\alpha}{2(\sqrt{\epsilon} + \sqrt{\beta\epsilon})} = n, \quad n = 1, 2, \dots \quad (37)$$

So the exact energy spectrum is given by

$$E_n = -\epsilon_n = -\frac{1}{4\beta} \left(1 - \sqrt{1 + \frac{2\alpha}{n}\sqrt{\beta}} \right)^2, \quad n = 1, 2, \dots, \quad (38)$$

without any free parameter obtained in [22]. Using equation (36), eigenfunctions also satisfy the following condition:

$$\int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \phi(p) dp = \mathcal{A} \frac{2\epsilon}{\alpha} \sin \left[\frac{\pi\alpha}{2\sqrt{\epsilon}(1 + \sqrt{\beta\epsilon})} \right] = 0. \quad (39)$$

It is worth mentioning that to derive the energy spectrum (38) we did not use any explicit assumption about the actual nature of c . The energy spectrum is based on the wave function (32) and the Hermiticity condition (28) where both relations are obtained regardless of the fact that c is a linear functional or a constant. Indeed our result can be used as a check for the validity of each assumption. Since the unitary transformation leads to the dependence of the energy spectrum on arbitrary phase, it implies that c is a constant in agreement with the proof presented after equation (24).

To find the domains of the Hamiltonian and its adjoint, since the operator $P^2 = \tan^2(\sqrt{\beta}p)/\beta$ is obviously a symmetric operator, we write the symmetricity condition for $1/X$ as

$$\left\langle \frac{1}{X} \varphi \middle| \phi \right\rangle = \left\langle \varphi \middle| \frac{1}{X} \phi \right\rangle, \quad (40)$$

where ϕ and φ belong to the domain of H and H^\dagger , respectively. This condition using the explicit expression for operator $1/X$ (17) can be rewritten as

$$\begin{aligned} i \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \phi(p) dp \int_{-\frac{\pi}{2\sqrt{\beta}}}^p \varphi^*(q) dq + c^* \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \phi(p) dp \\ = -i \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \varphi^*(p) dp \int_{-\frac{\pi}{2\sqrt{\beta}}}^p \phi(q) dq + c \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \varphi^*(p) dp. \end{aligned} \quad (41)$$

Now the identity

$$\int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} f(p) dp \int_{-\frac{\pi}{2\sqrt{\beta}}}^p g(q) dq = \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} g(p) dp \left[\int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} f(q) dq - \int_{-\frac{\pi}{2\sqrt{\beta}}}^p f(q) dq \right], \quad (42)$$

implies

$$i \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \phi(p) dp \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \varphi^*(q) dq + c^* \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \phi(p) dp - c \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \varphi^*(p) dp = 0. \quad (43)$$

Therefore, using (39) we obtain

$$\int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \varphi^*(p) dp = 0. \quad (44)$$

Now because of (39) and (44) the domains of H and H^\dagger coincide

$$\mathcal{D}(H) = \mathcal{D}(H^\dagger) = \left\{ \phi \in \mathcal{D}_{\max} \left(\frac{-\pi}{2\sqrt{\beta}}, \frac{+\pi}{2\sqrt{\beta}} \right); \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \phi(p) dp = 0 \right\}, \quad (45)$$

and the Hamiltonian is rendered a true self-adjoint operator [26].

3.1. Single-valuedness criteria

The requirement of single-valuedness of the eigenfunctions (32) leads to the following quantization condition:

$$\frac{\alpha}{2\sqrt{\epsilon}(1 - \beta\epsilon)} = m, \quad m = 1, 2, \dots \quad (46)$$

Note that unlike [22] there is no other term in (32) that influences the single-valuedness criteria. But the eigenfunctions obeying quantization condition (46) do not satisfy (39). Comparison between the two quantization conditions (37) and (46) shows that

$$m = \frac{n}{1 - \sqrt{\beta\epsilon}}. \quad (47)$$

So the single-valuedness criteria of the eigenfunctions ($m \in \text{Integers}$) is only valid at the limit $\beta \rightarrow 0$, i.e., the absence of GUP. In other words, in this case, the “single-valuedness” criteria can be considered as an emergent condition rather than a fundamental one.

3.2. Perturbative approach

By expanding the energy spectrum in terms of the GUP parameter we find

$$E_n = -\frac{\alpha^2}{4n^2} + \frac{\alpha^3}{4n^3}\sqrt{\beta} - \frac{5\alpha^4}{16n^4}\beta + \mathcal{O}(\beta^{3/2}), \quad n = 1, 2, \dots \quad (48)$$

So the first correction to the energy spectrum is proportional to $\sqrt{\beta}$. This result can be also understood from the perturbative study of this problem. Consider the GUP-corrected Hamiltonian to first-order of the deformation parameter

$$H \simeq p^2 - \frac{\alpha}{x} + (2/3)\beta p^4. \quad (49)$$

The evaluation of the first-order energy spectrum leads to

$$E_n = E_n^0 + \Delta E_n, \quad (50)$$

where E_n^0 are unperturbed energy eigenvalues and ΔE_n are given by

$$\Delta E_n = \frac{2}{3}\beta \langle \phi_n^0(p) | p^4 | \phi_n^0(p) \rangle. \quad (51)$$

Here $\phi_n^0(p)$ are solutions of (49) with $\beta = 0$. Also equation (34) shows that at this limit the probability density in momentum space takes the following form

$$|\phi_n^0(p)|^2 = \frac{2\epsilon^{3/2}}{\pi(p^2 + \epsilon)^2} = \frac{4\alpha^3 n}{\pi(\alpha^2 + 4n^2 p^2)^2}. \quad (52)$$

So the right hand side of (51) diverges that would explain the $\sqrt{\beta}$ term in (48): the integral is linearly divergent at large p , and since the natural momentum scale is $1/\sqrt{\beta}$ the net result is of order $\beta \times 1/\sqrt{\beta}$ which gives $\sqrt{\beta}$ and this divergent behavior also comes from all higher moments $\langle p^{2n} \rangle$ showing that expansion around $\beta = 0$ is not analytic.

4. Quasiposition representation

Following [9] we define the maximal localization states $|\phi_\xi^{\text{ML}}\rangle$ with the following properties:

$$\langle \phi_\xi^{\text{ML}} | X | \phi_\xi^{\text{ML}} \rangle = \xi, \quad (53)$$

and

$$\Delta X_{|\phi_\xi^{\text{ML}}\rangle} = (\Delta X)_{\min} = \hbar \sqrt{\beta}. \quad (54)$$

These states also satisfy

$$\left(X - \langle X \rangle + \frac{\langle [X, P] \rangle}{2(\Delta P)^2} (P - \langle P \rangle) \right) |\phi\rangle = 0, \quad (55)$$

where $\langle [X, P] \rangle = i\hbar (1 + \beta(\Delta P)^2 + \beta\langle P \rangle^2)$. So in momentum space the above equation takes the form

$$\left[i\hbar \frac{\partial}{\partial p} - \langle X \rangle + i\hbar \frac{1 + \beta(\Delta P)^2 + \beta\langle P \rangle^2}{2(\Delta P)^2} \left(\frac{\tan(\sqrt{\beta}p)}{\sqrt{\beta}} - \langle P \rangle \right) \right] \phi(p) = 0, \quad (56)$$

which has the solution

$$\begin{aligned} \phi(p) = \mathcal{N} \exp & \left[\left(-\frac{i}{\hbar} \langle X \rangle + \frac{1 + \beta(\Delta P)^2 + \beta\langle P \rangle^2}{2(\Delta P)^2} \langle P \rangle \right) p \right. \\ & \left. + \left(\frac{1 + \beta(\Delta P)^2 + \beta\langle P \rangle^2}{2(\Delta P)^2} \right) \frac{\ln [\cos(\sqrt{\beta}p)]}{\beta} \right]. \end{aligned} \quad (57)$$

To find the absolutely maximal localization states we need to choose the critical momentum uncertainty $\Delta P = 1/\sqrt{\beta}$ that gives the minimal length uncertainty and take $\langle P \rangle = 0$, i.e.,

$$\phi_\xi^{\text{ML}}(p) = \mathcal{N} \cos(\sqrt{\beta}p) e^{\frac{-ip\xi}{\hbar}}, \quad (58)$$

where the normalization factor is given by

$$\mathcal{N} = \sqrt{\frac{2\sqrt{\beta}}{\pi}}. \quad (59)$$

It is straightforward to check that $\phi_\xi^{\text{ML}}(p)$ exactly satisfies (53) and (54). Because of the fuzziness of space, these maximal localization states are not mutually orthogonal.

$$\langle \phi_{\xi'}^{\text{ML}} | \phi_\xi^{\text{ML}} \rangle = \mathcal{N}^2 \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} dp \cos^2(\sqrt{\beta}p) e^{\frac{-ip(\xi-\xi')}{\hbar}} = \frac{8\beta^{3/2}\hbar^3}{\pi} \frac{\sin\left[\frac{\pi(\xi-\xi')}{2\hbar\sqrt{\beta}}\right]}{(\xi-\xi')^3 - 4\beta\hbar^2(\xi-\xi')}. \quad (60)$$

To find the quasiposition wave function $\psi(\xi)$, we define

$$\psi(\xi) \equiv \langle \phi_\xi^{\text{ML}} | \phi \rangle, \quad (61)$$

where in the limit $\beta \rightarrow 0$ it goes to the ordinary position wave function $\psi(\xi) = \langle \xi | \phi \rangle$. Now the transformation of the wave function in the momentum representation into its counterpart quasiposition wave function is

$$\psi(\xi) = \mathcal{N} \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} dp \cos\left(\sqrt{\beta}p\right) e^{\frac{ip\xi}{\hbar}} \phi(p). \quad (62)$$

Although, regardless of energy, all wave functions in position space vanish at the origin for $\beta = 0$, i.e. $\psi^0(0) = 0$, in the presence of the minimal length, the quasiposition wave functions do not vanish generally at the origin where can be checked by numerical evaluation of (62).

5. Coordinate representation

The eigenfunctions of the position operator in momentum space are given by the solutions of the eigenvalue equation

$$X u_x(p) = x u_x(p), \quad (63)$$

where $u_x(p) = \langle p | x \rangle$. The normalized solutions are

$$u_x(p) = \sqrt{\frac{\sqrt{\beta}}{\pi}} \exp\left(-i\frac{p}{\hbar}x\right), \quad (64)$$

Now using (11) we find the wave function in coordinate space as

$$\eta(x) = \sqrt{\frac{\sqrt{\beta}}{\pi}} \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} e^{\frac{ipx}{\hbar}} \phi(p) dp. \quad (65)$$

Note that since the eigenfunctions of the position operator satisfy the zero uncertainty relation, i.e. $\Delta X_{|x\rangle} = 0$, $\eta(x)$ is not the physical wave function of the system. However, the generalized Schrödinger equation in coordinate space has a simple structure and $\eta(x)$ can be also considered as an intermediate solution.

In coordinate space, the wave function at the origin is given by

$$\eta(0) = \sqrt{\frac{\sqrt{\beta}}{\pi}} \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \phi(p) dp. \quad (66)$$

Now because of (39) it is rendered zero, namely

$$\eta(x)\Big|_{x=0} = 0. \quad (67)$$

So the coordinate space wave functions satisfy the Dirichlet boundary condition as well as in ordinary quantum mechanics [14].

6. Semiclassical approach

The energy spectrum can be also found using the Bohr-Sommerfeld quantization condition

$$\oint p dx = 2n\pi, \quad n = 1, 2, \dots \quad (68)$$

The corresponding classical Hamiltonian to this system is

$$H(x, p) = \frac{\tan^2(\sqrt{\beta}p)}{\beta} - \frac{\alpha}{x}. \quad (69)$$

Since the Hamiltonian is conserved, i.e. $H(x, p) = E$, we can express x as a function of p

$$x = \frac{\alpha\beta}{\tan^2(\sqrt{\beta}p) - \beta E}, \quad (70)$$

and use the identity $\oint p dx = -\oint x dp$. When the particle leaves the origin in positive direction p changes from $+\frac{\pi}{2\sqrt{\beta}}$ to 0 and when it returns to the origin in negative direction p changes from 0 to $-\frac{\pi}{2\sqrt{\beta}}$. So $-\oint x dp = \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} x dp$ and for the negative energy bound states we find

$$2n\pi = \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \frac{\alpha\beta}{\tan^2(\sqrt{\beta}p) - \beta E} dp = \frac{\pi\alpha}{\sqrt{\epsilon} + \sqrt{\beta\epsilon}}, \quad (71)$$

which exactly agrees with the quantum mechanical result (37).

7. WKB approximation

To check the validity of the Bohr-Sommerfeld quantization rule for this modified quantum mechanics, let us write the first-order generalized Schrödinger equation corresponding to the Hamiltonian

$$H(x, p) = p^2 + \frac{2}{3}\beta p^4 + V(x). \quad (72)$$

as

$$-\hbar^2 \frac{\partial^2 \psi(x)}{\partial x^2} + \frac{2}{3}\hbar^4 \beta \frac{\partial^4 \psi(x)}{\partial x^4} + V(x)\psi(x) = E\psi(x), \quad (73)$$

and take

$$\psi(x) = e^{i\Phi(x)}, \quad (74)$$

where $\Phi(x)$ can be expanded as a power series in \hbar in the semiclassical approximation i.e.

$$\Phi(x) = \frac{1}{\hbar} \sum_{n=0}^{\infty} \hbar^n \Phi_n(x). \quad (75)$$

So we have

$$\frac{\partial^2 \psi(x)}{\partial x^2} = -(\Phi'^2 - i\Phi'')\psi(x), \quad (76)$$

$$\frac{\partial^4 \psi(x)}{\partial x^4} = (\Phi'^4 - 6i\Phi'^2\Phi'' - 3\Phi''^2 - 4\Phi''' \Phi' + i\Phi''')\psi(x), \quad (77)$$

where Φ' indicates the derivative of Φ with respect to x . To the zeroth-order ($\Phi(x) \simeq \Phi_0(x)/\hbar$) and for $\hbar \rightarrow 0$ we obtain

$$\Phi_0'^2 + \frac{2\beta}{3}\Phi_0'^4 = E - V(x). \quad (78)$$

Now the comparison with Eq. (72) shows $\Phi'_0 = p$ and consequently

$$\psi(x) \simeq \exp \left[\frac{i}{\hbar} \int p \, dx \right], \quad (79)$$

which is the usual zeroth-order WKB wave function obeying the Bohr-Sommerfeld quantization rule. The generalization of this result to higher order perturbed Hamiltonian and the non-perturbative Hamiltonian (69) is also straightforward. Indeed, the agreement between the exact and semiclassical results is the manifestation of the validity of the Bohr-Sommerfeld quantization rule in this modified quantum mechanics [27].

8. Conclusions

In this paper, we found exact energy eigenvalues and eigenfunctions of the one-dimensional hydrogen atom by requiring the self-adjointness of the Hamiltonian. The energy spectrum is based on two main relations. The first is the differential equation in momentum space (32) and the second is the Hermiticity condition (28). In Ref. [22] the differential equation is solved in momentum space using KMM representation. However, the Hermiticity condition was not taken into account. Although the formal expression of the solutions are different, the value of c is rendered to be similar to (33), namely [22]

$$c = \sqrt{\frac{2}{\pi}} \frac{\epsilon^{\frac{3}{4}}}{\alpha} \frac{1 + \sqrt{\beta\epsilon}}{\sqrt{1 + 2\sqrt{\beta\epsilon}}} \exp \left[\frac{i\alpha\pi}{2(\sqrt{\epsilon} + \sqrt{\beta\epsilon})} \right]. \quad (80)$$

So if instead of the weaker condition (40), we apply the Hermiticity condition (28) to the results of Ref. [22], we recover the correct energy spectrum without any free parameter. As stated before, the Hermiticity condition is hold whether we take c to be constant or linear functional. However, the algebraic structure of X^{-1} [see e.g. (18)-(20)] and the behavior of the solutions under the unitary transformation do not support the latter assumption.

After finding the maximal localization states we obtained the quasiposition wave functions and showed that unlike the coordinate space solutions, they do not vanish generally at the origin. Moreover, we indicated that the WKB approximation is valid in this deformed algebra and the semiclassical energy spectrum exactly coincides with the quantum mechanical results. It is also shown that the single-valuedness criteria is an emergent condition in ordinary quantum mechanics.

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